Wave analysis of a string bowed to anomalous low frequencies

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ABSTRACT

A bowed string bears the possibilities of producing steady state oscillations in numerous ways and with a great variety of fundamental frequencies. By means of computer simulations, this paper will show how a string how a string can be forced to vibrate with lower fundamental frequencies than the natural first mode frequency, f_0, of the system. These frequencies are not usually true "subharmonics", since they are not simply related to f_0. The triggering of slip and stick is often determined by a combination of transverse and torsional waves and their respective reflections at the bow and the endpoints of the string. The analysis discusses how the bowing position and the torsional wave velocity influence the possible lengths of periods.

INTRODUCTION

In a companion article in this issue, Hanson et al. [1] show how violin strings can be forced to oscillate at anomalous low frequencies (ALF) when bowing with high bow force (bow "pressure"). Graphs of the string movements demonstrate a number of pitch drop intervals with the string oscillating at steady state with good stability.

In order to understand the slip-stick triggering mechanism involved, a computer program capable of simulating transverse and torsional movements of the string has been employed. The programmable parameters are: wave impedances and velocities, reflection functions, bow compliance, bow velocity and "pressure" as functions of time, and their interaction with the hyperbolic friction characteristic of the string and rosined bow. The program, as well as the present analysis, is based on D'Alembert's solution [2] to the wave equation, where the displacements of waves moving in opposite directions are shifted with time. The solution in the continuous form reads:

$$\eta(x,t) = \eta_+(x - ct) + \eta_-(x + ct) \qquad (1)$$

where $\eta_{(+/-)}$ = (partial) string displacement; c = the propagating speed of the wave; x = position on the string; t = time.

Two sets of eq. (1) are used: one for transverse-and one for torsional waves. In the simulation program, these interact only at the point of bowing, and only through frictional force. Dispersion and losses are carried out by reflection functions "at the string terminations". In general, the concepts of string simulation developed by McIntyre, Woodhouse and Schumacher [3][4][5][6][7] have been adopted.

SIMULATION PARAMETERS

Signal: throughout this paper the term "signal" means derivative of (partial) displacement with respect to time, i.e., $\delta\eta_{(+/-)}(x)/\delta t$. The following fixed parameters are used for the simulations of the strings A and B as presented in the graphs:

transverse nut reflection (response to a signal impulse at t = 0): $R_1(t) = -1/[\sigma\sqrt{(2\pi)}]\exp[-0.5(t/\sigma)^2]$, i.e., a "Gaussian function" (for convolution), where $\sigma = 0.00357T_0$, and $T_0 = 1/f_0$ = the natural transverse oscillating period of the string. In the discrete version, the normalized sum of the function equals -1;

transverse bridge reflection (response to a signal step at t = 0): $R_2(t) = 0.077\exp(-t/\tau_1)-1$, i.e., a "Cremer function" [8], where $\tau_1 = 0.049\ T_0$;

torsional nut and bridge reflections (responses to a signal step at t = 0) respectively: $R_3(t) = R_4(t) = \exp(-t/\tau_2)-1$, where $\tau_2 = 0.016\ T_0$;

transverse and torsional wave velocities, respectively: c_{TRV} and c_{TOR}. In the figures showing string A, the ratio $c_{TRV}/c_{TOR} = \zeta = 0.3$, while in figures showing string B, $\zeta = 0.213$ (which gives torsional first mode frequencies of $3.2f_0$ and $4.15f_0$ respectively when accounting for the phase shifts of $R_3(t)$ and $R_4(t)$);

return time of transverse reflected waves (with paths: bow/nut/bow and bow/bridge/bow) respectively: $t_1 = 2(1-\beta)L/c_{TRV}$ and $t_2 = 2\beta L/c_{TRV}$, where $\beta = 0.045$ = the distance from the bridge to the bowing point divided by the total string length (L). A normal slipping interval would be equal to t_2;

ratio of torsional and transverse impedances: $\Omega = Z_{TRV}/Z_{TOR} = 0.65$, where Z_{TRV} and Z_{TOR} = the characteristic wave resistance of transverse and torsional waves, respectively. To facilitate comparison while emphasizing the effects of torsion, the chosen value of Ω is kept constant throughout the simulations, regardless of the wave velocity ratio ζ;

impedance of the bow is infinite, i.e., the bow is not compliant; the bow excites the string in one single point;

the frictional characteristic of the rosin is hyperbolic, and matches values used by Schumacher/Woodhouse in ref [9]. In their terminology $v_{mid} = -1.5v_b$, indicating a friction coefficient that has fallen half way to its asymptotic minimum value for a relative bow/string velocity of 2.5 times the bow velocity;

Q-values of the two strings, A and B are shown in [Figure 1].

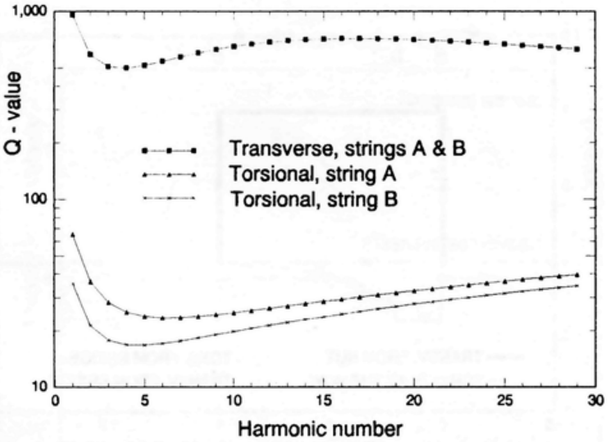

Figure 1: Transverse and torsional damping of free oscillations of the simulated strings A and B.

POTENTIAL FRICTIONAL FORCE

The frictional force (f_{ST}) between the non-compliant bow and the string during the *static* intervals may be expressed through the following equation (which can be derived from ref. [6], eqs. (B13) and (B14)):

$$f_{ST}(t) = 2Z_{CMB}\,[V_{BOW} - \sum v_{w(1-4)}\,(t)] \qquad (2)$$

where Z_{CMB} = the combined wave impedance of the string, i.e.,

$$(Z_{TRV}\ Z_{TOR})/(Z_{TRV} + Z_{TOR});$$

V_{BOW} = velocity of the bow;
$v_{W(i)}(t)$ = signal of partial wave i, arriving at the bow:
$v_{W1}(t)$ and $v_{w2}(t)$ = *transverse* signals propagating away from the nut and the bridge, respectively;
$v_{W3}(t)$ and $w_{W4}(t)$ = *torsional* signals [10] propagating away from the nut and the bridge, respectively. The sum of the four partial signals (i.e., velocities) gives the *surface velocity* the string would have taken at the point of bowing without friction.

It is convenient to refer to f_{ST} as *"potential frictional force"*, disregarding whether the friction be static or not. The value of $f_{ST}(t)$ expresses the frictional force that occurs as long as the limiting (static) frictional force (f_L) is greater than $f_{ST}(t)$. At the very moment $f_L < f_{ST}(t)$, the friction will change from static to sliding. Thus, $f_{ST}(t) - f_L$ expresses the margin for this change to happen, at any instant.

PULSE RESPONSES

To see how a velocity pulse is reflected and keeps transforming between transverse torsional waves, the following simulations were run: at t = 0, the string is released from an initial maximum *transverse* displacement (η) at the point of bowing, with decreasing displacement on both sides toward the bridge and the nut—similar to a pizzicato. (The string is pulled in positive bowing direction at its center: *no torsional displacement.*) During one "normal *slipping* interval", $0 \leq t < t_2$, there is no contact between the bow and the string. When $t \geq t_2$, contact between bow and string is established, and the limiting static friction is kept high enough to ensure a static grip. We thus have an initial transverse "velocity pulse" (V_{INI}) which is t_2 wide and $-\eta f_0/\beta(1 - \beta)$ high, functioning as a test signal, and can observe its reflections and transformations. When interpreting the graphs, it should be remembered that the D'Alembert equation gives partial velocities (signals) even for a string held with fixed displacement.

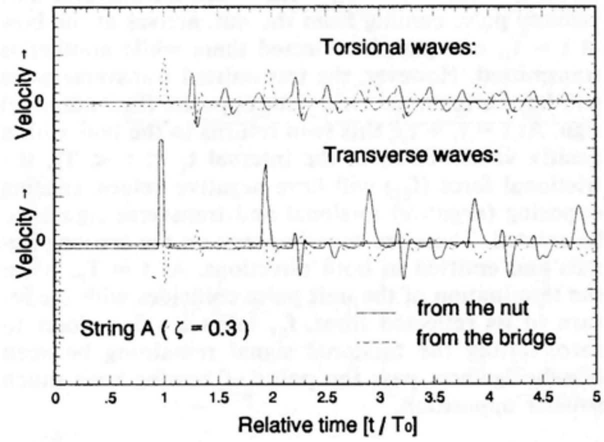

Figure 2: String A: transverse and torsional waves arriving at the bow after nut and bridge reflections of a velocity (signal) pulse V_{INI}.

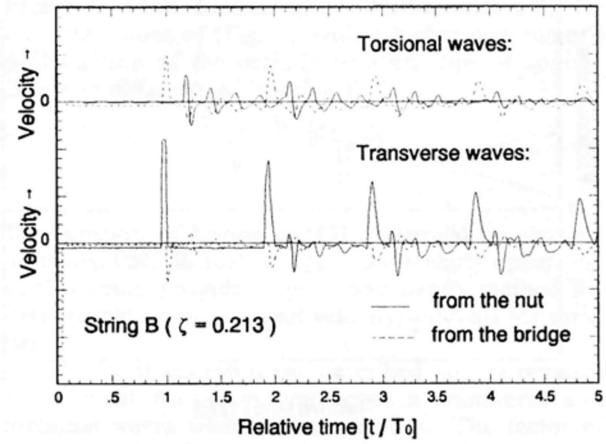

Figure 3: String B: transverse and torsional waves arriving at the bow after nut and bridge reflections of a velocity (signal) pulse V_{INI}.

In [Figures 2 and 3] (showing responses of the strings A and B, respectively) the bow velocity is zero. The first torsional wave of any dynamic significance is initiated at the time $t = t_1$ after the release of the string. This is the point of time at which the transverse wave the first time returns to the bow after having been reflected at the nut. Torsional waves are then excited by the string rolling up the fixed bow. (Torsional velocity is to be understood as string radius times radians per second, with the convention that the angle of twist is negative when the translational displacement is positive *on a fixed bow holding a static grip.* In the [Figs. 2 and 3], the scaling is equal to that of the transverse velocity, or signal, although drawn above for clarity.) This first torsional pulse will appear at the graphs at the time $t = t_1 + t_2\zeta$, reaching its maximum at $t = T_0 + t_2\zeta$, after several trips between the bridge and the bow.

See [Figure 4] for detail of how a (*positive*) torsional signal arriving at the bow, builds up in the interval $t_1 + t_2\zeta < t < T_0 + t_2\zeta$. When a transverse unit velocity pulse coming from the nut, arrives at the bow at $t = t_1$, one part is reflected there while another is transmitted. However, the transmitted transverse wave will have a torsional twin with opposite (i.e., negative) sign. At $t = t_1 + t_2\zeta$ this twin returns to the bow with a positiv value. Thus, in the interval $t_1 < t < T_0$, the frictional force (f_{ST}) will have negative values, causing opposing (*negative*) torsional and transverse signals to be created. These are superimposed on the arriving signals and emitted in both directions. At $t = T_0$, when the termination of the unit pulse coincides with the return of its reflected front, f_{ST} takes a value closer to zero, letting the torsional signal remaining between bow/bridge/bow pass the point of bowing with much smaller opposition.

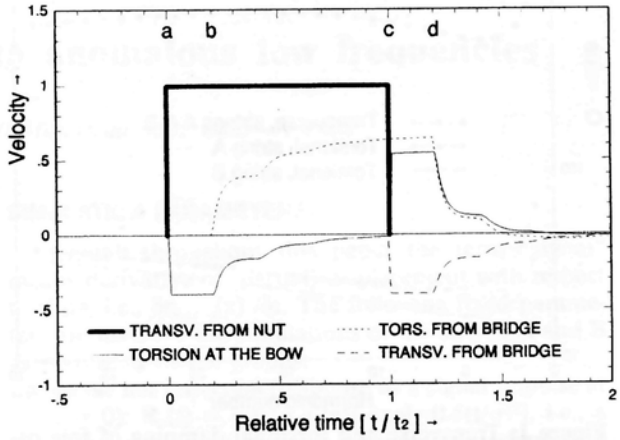

Figure 4: When a transverse unit velocity pulse (t_2 wide, and propagating away from the nut) arrives at the bow, a torsional pulse will build up after several reflections at between the bridge and the bow. If the unit pulse arrives at $t = t_1$, the torsional pulse arriving at the bow from the bridge, will reach its maximum at $T_0 + \zeta t_2$. Simulation parameters (with reduced losses): $R_2(t) = -\delta$; $R_4(t) = \exp(-t/\tau_2)-1$, where $\tau_2 = 0.0016\ T_0$; $\Omega = 0.65$; $\zeta = 0.200$. Certain points of time are marked for comparison with figures 2 and 3: $a = t_1$, $b = t_1 + \zeta t_2$, $c = T_0$, $d = T_0 + \zeta t_2$.

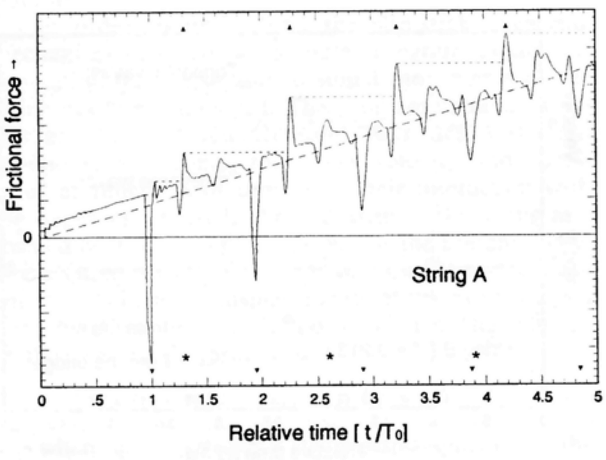

Figure 5: Frictional force on string A when the reflections of an initial velocity pulse, V_{INI}, are superimposed on the force originating from a bow pulling the string with a fixed velocity (dashed ramp). Force peaks caused by torsional nut reflections occur in intervals of approximately t_1 (marked ▲), starting at $t = T_0(1 + \zeta)$. These are potential triggers of slipping. The points of time for force reductions (marked ▾) are determined by transverse nut reflections of the initial string fly-back, and occur with the same intervals. Integer multiples of $T_0(1 + \zeta)$ are marked *, see text.

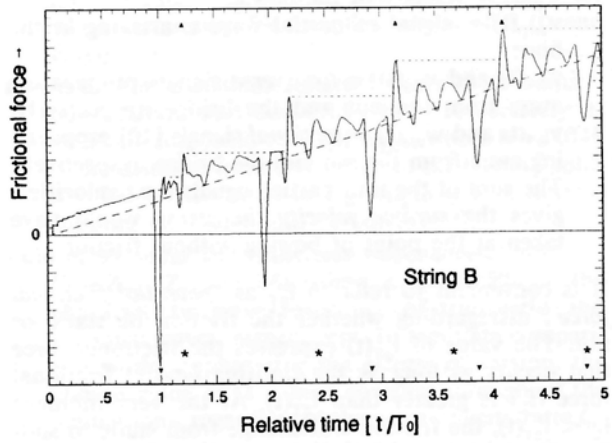

Figure 6: String B: V_{INI} and V_{BOW} are the same as for string A in [Figure 5]. Notice how the wave velocity ratio ($\zeta = 0.300$ for string A; $\zeta = 0.213$ for string B) determines the positions of force peaks on the time axis.

In the [Figures 5 and 6] the bow takes the velocity of $0.001\eta/s$ at the time $t \geq t_2$. The effect of the bow pulling the (reflecting) string with a constant velocity, is then superimposed on the reflections of V_{INI}, causing the average frictional force to increase with time. There are certain time spans where a release may take place, provided the limiting static frictional force is low enough. However, there are even larger time spans where release cannot happen, due to the fact that during these, the frictional force "falls in the shadow" of a preceding force peak of a higher value (indicated by horizontal, dashed lines). When comparing the two figures, one will appreciate the role of the wave velocity ratio, ζ, which is the only parameter changed between the two.

During a normal Helmholtz movement [11] of the string, the positive peak at $t/T_0 = 1$ will trigger a string release. If the limiting static frictional force is higher than the potential frictional force at $t = T_0$, the release will be delayed until a peak high enough does the trick. The next opportunity of triggering appears with the edge rising to the peak at $t = T_0(1 + \zeta)$, which is equal to $1.300T_0$ for string A, and $1.213T_0$ T_0 for B.

[Figures 7 a and b] show, respectively, waves arriving at the bow and frictional forces, as they appear during Helmholtz motion of string A.

TORSIONAL TRIGGERING OF ANOMALOUS LOW FREQUENCIES (ALF)

During a steady-state oscillation with delayed releases, the reflections of one string flyback will combine with the fading reflections of the previous string releases. Dependent on the interval between two slips, old and new reflections may combine to powerful peaks of f_{ST}, which ensure stable triggering, or they can lead to cancellation of the same. [Figures 8 a and b] show how string A forms new fundamental periods, which are stable and prolonged by 29.3%. This is somewhat more than *a major third* step down. For string B ([Figures 9 a and b]), the prolongation is about 20.4%, corresponding to about *a minor third* down. In both cases the triggering is stable, with reasonable "reserves" of f_{ST}, as compared to the limiting static frictional force (dashed horizontal line). Notice that all the figures in this article are drawn to the same scales with respect to their coordinates, for ease of comparison.

The bow force tolerances are less in [Figs. 8 and 9] than they are during the Helmholtz motion of [Fig. 7]. This is due to the partial cancellation that takes place if $nT_0(1 + \zeta) \approx (mt_1$ through $mt_1 + t_2)$ (where n = 1,2,3... and m = 1,2,3 ...). While the left side of this expression indicates integer multiples of the time of the first "torsional force maximum", the right side indicates time spans of "transverse force reductions". In [Figs. 5 and 6], torsional maxima and transverse minima are marked ▴ and ▾ respectively, while integer multiples of $1 + \zeta$ are marked *. The width of the force reduction is about t_2 when m = 1, but increases with higher values of m, due to dispersion and losses during the reflections. Concerning string A, the "soft cancellation" takes place near the time $t \approx 3.9T_0$ after a string release (i.e., * and ▾ falling close when n = 3, and m = 4). For string B, the corresponding values are $t \approx 4.85T_0$; n = 4, and m = 5. It was noticed during the simulations that cancellation was a greater problem for lower bow velocities than for higher. For example: when string B was bowed with 1/4 of the bow velocity used for [Figure 9], the typical sequence of periods would be: $[T_0\ (1 + \zeta)]$, $[T_0\ (1 + \zeta)]$, $[T_0\ (1 + \zeta)]$, $[T_0\ (1 + \zeta + t_1\zeta)]$ ..., with each fourth period additionally prolonged by $t_1\zeta$. This implies that the last triggering is initiated by a torsional wave arriving at the bow *after a second reflection at the nut.*

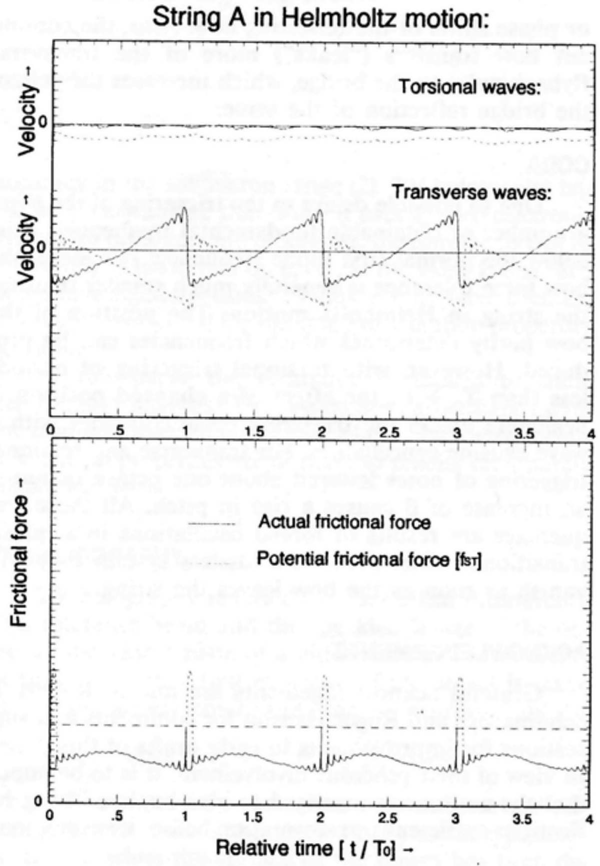

Figure 7: Waves arriving at the bow (a—upper) and frictional forces (b—lower graph) when string A is bowed to a normal Helmholtz motion. Spikes of the actual frictional force trigger string releases at t/T_0 = 1,2,3, etc. (With numeric simulation these are not usually recorded as f_L, this value not lasting till the conclusion of the time step: here they are *drawn* to full height.) The height of f_{ST} above the wavy act.fr.force, determines the bow force tolerance. The dashed horizontal line shows the *limiting static frictional force* applied for the simulation.

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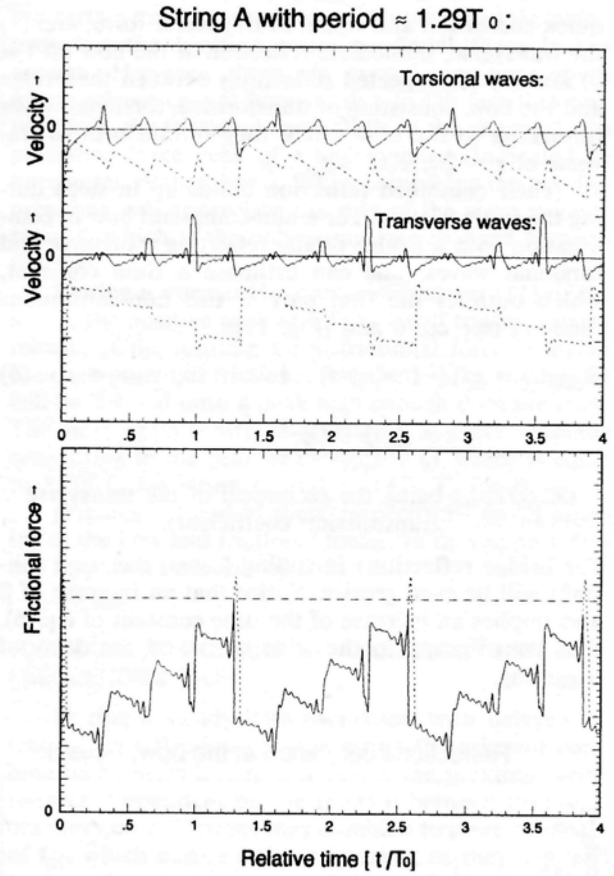

Figure 8: String A: when combining the bow velocity used for the simulations of [Fig. 7], with a higher bow force, a prolongation of the periods between slips of approx. 29.3% is obtained. The force peak at $t/T_0 = 1$ would have triggered a Helmholtz motion, had the limiting frictional force been lower.

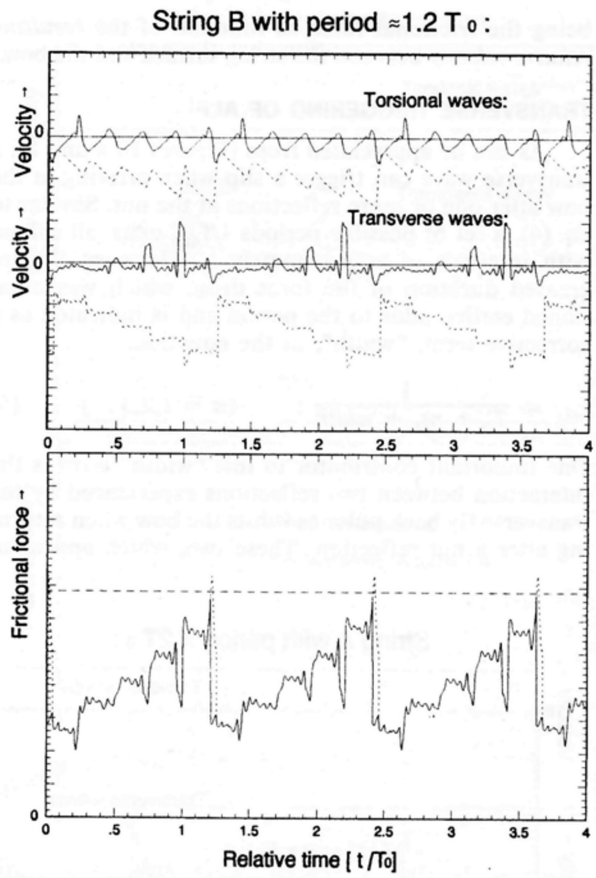

Figure 9: When combining the bow velocity used for the simulations of [Fig. 7], with a higher bow force, a prolongation of the periods between slips of approx. 20.4% is obtained with string B.

Due to the nonlinear behavior of the system, as well as the combinations of multiple reflections, it is difficult to give exact equations concerning prolongation, etc. But, from the expressions above, one can derive the following "thumb rules", which may show useful in some practical applications: For a stable, lowered frequency (f_{ALF}) triggered by torsional waves, the approximate wave velocity ratio (ζ) can be found through the equation:

$$\zeta \approx \frac{f_0}{f_{ALF}} - 1; \quad [f_0(1-\beta) > f_{ALF} > \frac{f_0}{2-\beta} = \frac{1}{T_0 + t_1}] \quad (3)$$

The best estimation of ζ is done if β is small, because then a narrow triggering pulse is created, and triggering bound to happen near to its peak, i.e., near $t = T_0(1 + \zeta)$. The frequencies most likely to occur as results of torsional triggering are:

$$f_{ALF} \approx \frac{1}{T_0(1+\zeta) + nt_1}; \quad (n = 0,1,2\ldots; \zeta > \frac{t_2}{t_1}) \quad (4)$$

The prospect of having eq. (3) confirmed by physical measurements on real strings is particularly appealing, as this could provide a quick and handy method for first estimation of torsional velocity—at least for some strings.

Torsional triggering, as described above, requires an efficient *transformation* between transverse and torsional waves when hitting the bow. The factor of transformation from transverse to torsional velocity [10] at a non-compliant bow is equal to $-Z_{CMB}/Z_{TOR}$, while the *transverse reflection* factor is $-Z_{CMB}/Z_{TRV}$, and the *transmission* coefficient Z_{CMB}/Z_{TOR}. Correspondingly, the transformation from torsional to transverse velocities is equal to $-Z_{CMB}/Z_{TRV}$, and the *torsional* reflection factor equal to $-Z_{CMB}/Z_{TOR}$, while the transmission coefficient is Z_{CMB}/Z_{TRV}. Hence, the chances of achieving a comparatively high, "torsional" f_{ST} increase as the value of Ω approaches unity. During sliding friction, partial signals (transverse and torsional) equal to $f_S/2Z_{TRV}$ and $f_S/2Z_{TOR}$ are superimposed on the partial signals arriving at the bow; "f_S"

being the frictional force as function of the *resulting relative velocity* between the string surface and the bow.

TRANSVERSE TRIGGERING OF ALF

As can be appreciated from [Figures 10 a and b], a transverse wave can trigger a slip when arriving at the bow after one or more reflections at the nut. Similar to eq. (4), a set of possible periods $1/f_{ALF}$ exits, all spaced with intervals of approximately t_1. However, the increased duration of the force drop, which was mentioned earlier, adds to the period and is indicated as a corrective term, "width", in the equation:

$$f_{ALF} \approx \frac{1}{T_0 + nt_1 + width}\,; \qquad (n = 1,2,3\,..) \qquad (5)$$

One important contributor to this "width" term is the interaction between two reflections experienced by the transverse fly back pulse as it hits the bow when returning after a nut reflection. These two, which appear in

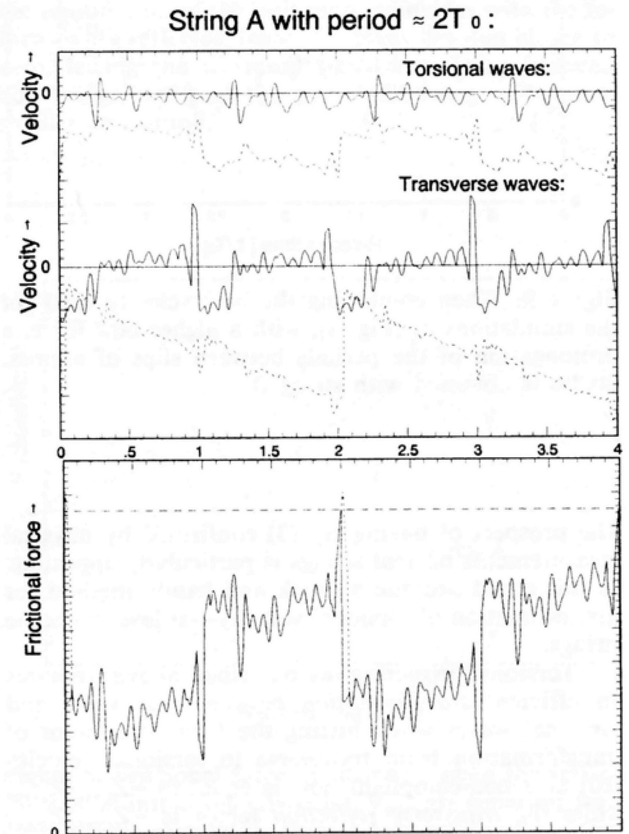

Figure 10: When applying a bow force four times higher than the one used for the Helmholtz motion of [Fig. 7], a period prolongation of approx. 100% is obtained. The triggering is here caused by transverse waves after nut reflections, as opposed to the [Figures 8 and 9] where torsional waves did the triggering.

quick succession and rejoin in transverse form, are: (1) the transverse, immediate reflection at the bow (at t = 0) and (2) the repeated reflections between the bridge and the bow, consisting of transformed, torsional waves (returning to the bow where they retransform to traverse at $t = \zeta t_2, 2\zeta t_2, 3\zeta t_2, ...$).

Their combined reflection builds up in steps during the static friction. For a non-compliant bow in combination with a bridge totally reflecting transverse and torsional waves, one can estimate a time constant, which outlines the first part of this non-continuous function (see eq. 6 and [Fig. 11]):

$$R_{CMB}(t) = \exp[-(t+\zeta t_2)/\tau] - 1 \qquad (0_+ \leq t < t_2) \quad (6)$$

$$\textit{where:} \qquad \tau = \frac{\zeta t_2}{\ln(Z_{TOR}/Z_{CMB})}$$

(Z_{TOR}/Z_{CMB} being the reciprocal of the transverse transmission coefficient).

For bridge reflections including losses, the time constant will be even greater. Notice that an increase of β also implies an increase of the time constant of eq. (6), thus contributing to the wide term and the delay of triggering.

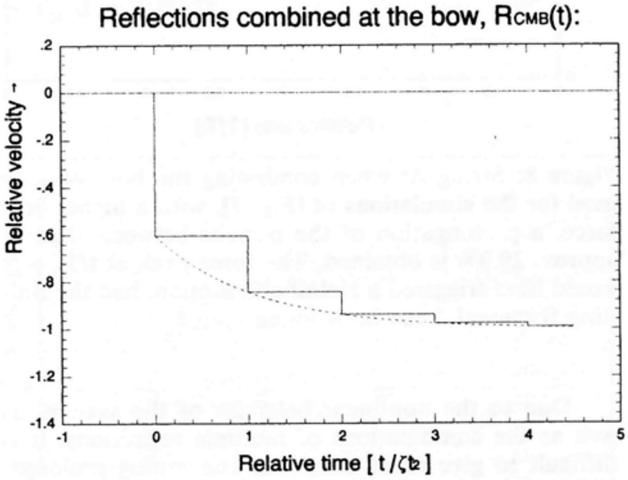

Figure 11: Combined reflections at the bow. The dotted line shows the function $R_{CMB}(t)$, which outlines the steps of a double reflection (solid line) (Ω = 0.65, see text).

BOW COMPLIANCE

All the above analyses are based on a non-compliant bow. With bow compliance, the picture becomes more blurred: simulations show that the expression "t_1" of equations (4) and (5) should be substituted with "$t_1 + d$", where "d" is a small value indicating the delay in the (peak) signal build up, caused by the losses or phase shifts of the reflecting bow. Also: the compliant bow transfers ("leaks") more of the transverse flyback pulse to the bridge, which increases the role of the bridge reflection of the same.

CODA

Due to possible delays in the triggering of the slips, a number of obtainable fundamental frequencies exist below the normal first mode frequency. For these, the bow force tolerance is generally much smaller than for the string in Helmholtz motion. The position of the bow partly determines which frequencies can be produced. However, with torsional triggering of periods less than $T_0 + t_1$, the effect of a changed position is negligible unless the triggering pulse coincides with a wave causing cancellation. For transverse and torsional triggering of notes lowered about one octave or more, an increase of β causes a rise in pitch. All these frequencies are results of forced oscillations in a quasi-transitional state: even on a lossless system they will vanish as soon as the bow leaves the string.

ACKNOWLEDGEMENTS

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